Comment on "Soliton ratchets induced by excitation of internal modes"

Niurka R. Quintero*
Departamento de Física Aplicada I, E. U. P., Universidad de Sevilla, Virgen de África 7, 41011 Sevilla, Spain and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, 18071 Granada, Spain

Bernardo Sánchez-Rey
Nonlinear Physics Group, University of Seville, Spain

Jesús Casado-Pascual
Universidad de Sevilla, Física Teórica, Apartado de Correos 1065, Sevilla 41080, Spain
(Dated: February 8, 2008)

Very recently Willis et al. [Phys. Rev. E **69**, 056612 (2004)] have used a collective variable theory to explain the appearance of a nonzero energy current in an ac driven, damped sine-Gordon equation. In this comment, we prove rigorously that the time-averaged energy current in an ac driven nonlinear Klein-Gordon system is strictly zero.

PACS numbers: 05.45.Yv, 05.45.-a, 05.60.Cd, 63.20.Pw, 02.30.Jr
Keywords:

Recently several papers have been published trying to understand soliton ratchets (see, for example, Refs. [1, 2, 3, 4, 5, 6] and for a recent review chapter 9 in Ref. [7], pp. 343–364). This phenomenon is a generalization of the ratchet effect [8] to spatially extended systems, and manifests as a unidirectional motion of a soliton induced by zero-average forces. A paradigmatic example is the driven, damped nonlinear Klein-Gordon equation:

$$\phi_{,tt}(x,t) - \phi_{,xx}(x,t) = -U'\left[\phi(x,t)\right] + f(t) - \beta\phi_{,t}(x,t), \tag{1}$$

where $g_{,z} = \partial g/\partial z$, $f(t)$ is a periodic field with period T and zero time-average [i.e., $1/T\int_0^T dt\, f(t) = 0$], $\beta > 0$ is the dissipation parameter determining the inverse relaxation time in the system, and $U'(z)$ is the derivative with respect to z of the potential $U(z)$. In this comment, we will assume that the potential $U(z)$ is periodic with period λ, and presents minima at $z_j = z_0 + j\lambda$, with $j \in \mathbb{Z}$. The ac driven, damped sine-Gordon equation considered in Ref. [6] is a particular case of this more general problem, with $U(z) = 1 - \cos(z)$ and

$$f(t) = -\left[\epsilon_1 \cos(\omega t) + \epsilon_2 \cos(2\omega t + \theta)\right]. \tag{2}$$

To fully specify the mathematical problem, the partial differential equation (1) must be amended by both initial conditions for $\phi(x,0)$ and $\phi_{,t}(x,0)$, and boundary conditions for $\lim_{x\to\pm\infty}\phi(x,t)$. Several boundary conditions can be imposed to have a well-posed boundary value problem. For instance, in the absence of the periodic field $f(t)$, it is possible to choose the fixed boundary conditions: $\lim_{x\to+\infty}\phi(x,t) = z_l$ and $\lim_{x\to-\infty}\phi(x,t) = z_m$. In the presence of $f(t)$, the fixed boundary conditions become incompatible with Eq. (1), and they are usually replaced by the aperiodic boundary conditions:

$$\lim_{x\to+\infty}\phi(x,t) = \lim_{x\to-\infty}\phi(x,t) + \lambda Q, \tag{3}$$

$$\lim_{x\to+\infty}\phi_{,x}(x,t) = \lim_{x\to-\infty}\phi_{,x}(x,t), \tag{4}$$

where $Q \in \mathbb{Z}$ is the so-called topological charge. The discrete version of these periodic boundary conditions are also the most used in the numerical solution of Eq. (1) (see, for example, Refs. [1] and [5]).

It can be derived from the continuity equation that the energy current density generated by the field $\phi(x,t)$ in the absence of damping and external forcing is given by $j(x,t) = -\phi_{,t}(x,t)\phi_{,x}(x,t)$ and, consequently, the energy current reads

$$J(t) = -\int_{-\infty}^{+\infty} dx\ \phi_{,x}(x,t)\phi_{,t}(x,t). \tag{5}$$

The time-averaged energy current $\langle J\rangle$ is defined as the limit

$$\langle J\rangle = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau dt\ J(t). \tag{6}$$

In Ref. [1], it has been proved by symmetry considerations that a *necessary* condition for the appearance of a non-vanishing time-averaged energy current is that either the potential presents broken spatial symmetry, or the field $f(t)$ violates the symmetry property

$$f\left(t + \frac{T}{2}\right) = -f(t), \tag{7}$$

or both simultaneously. Following this idea, a collective variable approach has been developed in Ref. [6] for the ac driven, damped sine-Gordon equation with a field of the form (2) which leads to a non-vanishing time-averaged energy current. The purpose of this comment is to prove that the time-averaged energy current, $\langle J\rangle$, of a

*Electronic address: niurka@euler.us.es

driven, damped nonlinear Klein-Gordon equation of the form (1) is necessarily zero and, consequently, the above mentioned results in Ref. [6] must be erroneous.

To prove that $\langle J \rangle = 0$, firstly we will obtain an ordinary differential equation for the energy current $J(t)$. In order to do that, we differentiate with respect to time Eq. (5), resulting

$$\dot{J}(t) = -\int_{-\infty}^{+\infty} dx \left[\phi_{,xt}(x,t)\phi_{,t}(x,t) + \phi_{,x}(x,t)\phi_{,tt}(x,t)\right]. \tag{8}$$

By making use of Eq. (1) in the second term on the right-hand side of the above expression, it is straightforward to write it in the form

$$\begin{aligned} \dot{J}(t) &= -\frac{1}{2}\left\{\lim_{x\to+\infty}\left[\phi_{,t}(x,t)\right]^2 - \lim_{x\to-\infty}\left[\phi_{,t}(x,t)\right]^2\right\} \\ &\quad -\frac{1}{2}\left\{\lim_{x\to+\infty}\left[\phi_{,x}(x,t)\right]^2 - \lim_{x\to-\infty}\left[\phi_{,x}(x,t)\right]^2\right\} \\ &\quad + \lim_{x\to+\infty} U\left[\phi(x,t)\right] - \lim_{x\to-\infty} U\left[\phi(x,t)\right] \\ &\quad -\beta J(t) - \left[\lim_{x\to+\infty}\phi(x,t) - \lim_{x\to-\infty}\phi(x,t)\right] f(t). \end{aligned} \tag{9}$$

Differentiating Eq. (3) with respect to t, it is easy to see that the first term between brace brackets on the right-hand side of Eq. (9) is equal to zero. From Eq. (4), it follows that the second term between brace brackets on the right-hand side of Eq. (9) is also equal to zero. The two terms of Eq. (9) containing $U[\phi(x,t)]$ also cancel each other due to the boundary condition (3) and the periodicity of $U(z)$. Thus, from Eq. (3) we finally obtain

$$\dot{J}(t) = -\beta J(t) - \lambda Q f(t). \tag{10}$$

It is important to emphasize that Eq. (10) is a direct consequence of Eq. (1) and the boundary conditions (3) and (4). Therefore, it is an *exact* result valid for any periodic potential $U(z)$ of the type described in the paragraph below Eq. (1), and any external field $f(t)$. Equation (10) appears in Ref. [6] as an *approximate* result obtained after neglecting the dressing due to phonons.

The general solution of Eq. (10) is

$$J(t) = J(0)e^{-\beta t} - \lambda Q \int_0^t dt' \, e^{-\beta(t-t')} f(t'), \tag{11}$$

and making use of the definition of the time-averaged energy current in Eq. (6), it results

$$\begin{aligned} \langle J \rangle &= \lim_{\tau\to+\infty}\left\{\frac{J(0)}{\beta\tau}(1-e^{-\beta\tau}) - \frac{\lambda Q}{\beta\tau}\int_0^\tau dt\, f(t) \right. \\ &\quad \left. + \frac{\lambda Q}{\beta\tau}\int_0^\tau dt\, f(t)e^{-\beta(\tau-t)}\right\}. \end{aligned} \tag{12}$$

The first limit in the above expression is obviously equal to zero. The second one is also equal to zero as the external field is periodic with zero time-average. The integral appearing in Eq. (12) can be bounded using the fact that

$$\begin{aligned} \left|\int_0^\tau dt\, f(t)e^{-\beta(\tau-t)}\right| &\le \int_0^\tau dt\, |f(t)|\, e^{-\beta(\tau-t)} \\ &\le \frac{f_m}{\beta}\left(1-e^{-\beta\tau}\right), \end{aligned} \tag{13}$$

where f_m is an upper bound of $|f(t)|$ and, thus, the third limit in Eq. (12) is also equal to zero. *We conclude that* $\langle J \rangle = 0$ *for any periodic potential* $U(z)$ *of the type described in the paragraph below Eq. (1), and any bounded, zero time-averaged periodic field* $f(t)$.

To corroborate this result, we have solved numerically the sine-Gordon equation for the case of the biharmonic field in Eq. (2). We have considered the initial conditions for $\phi(x,0)$ and $\phi_{,t}(x,0)$ corresponding to a static kink-like solution of the unperturbed sine-Gordon equation [i.e., $f(t) = \beta = 0$], so that, $\lambda = 2\pi$, $Q = 1$, and $J(0) = 0$. For this particular choice, the energy current in Eq. (11) reads

$$\begin{aligned} J(t) &= \frac{2\pi\epsilon_1}{\beta^2+\omega^2}\left[-\beta e^{-\beta t} + \beta\cos(\omega t) + \omega\sin(\omega t)\right] \\ &\quad + \frac{2\pi\epsilon_2}{\beta^2+4\omega^2}\left[-e^{-\beta t}(\beta\cos\theta + 2\omega\sin\theta)\right. \\ &\quad \left. + \beta\cos(2\omega t+\theta) + 2\omega\sin(2\omega t+\theta)\right]. \end{aligned} \tag{14}$$

Figure 1 shows the perfect agreement between the above expression and the numerical simulation of the sine-

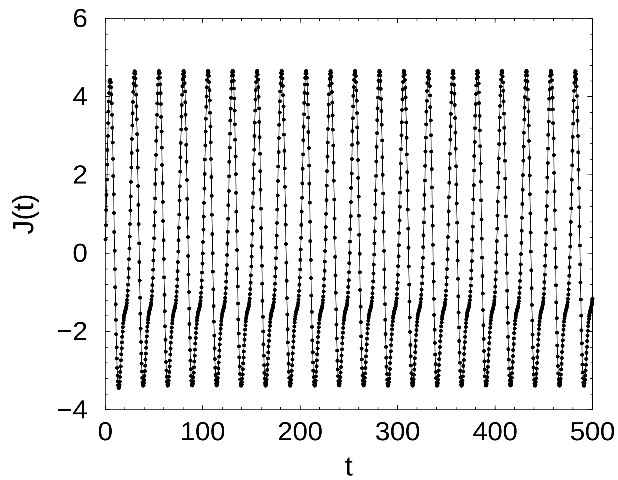

FIG. 1: Evolution of the energy current $J(t)$ corresponding to the sine-Gordon equation with the biharmonic field in Eq. (2) and the initial conditions corresponding to a static kink-like solution of the unperturbed sine-Gordon equation [i.e., $f(t) = \beta = 0$]. The parameter values are the same as in Fig. 3 of Ref. [6]: $\beta = 0.15$, $\epsilon_1 = 0.16$, $\epsilon_2 = \epsilon_1/\sqrt{2}$, $\omega = 0.25$, and $\theta = 1.61 - \pi$. The points represents the results obtained by solving numerically the sine-Gordon equation and the solid line the analytical expression in Eq. (14). The agreement is obviously excellent since Eq. (14) is exact. The time-averaged energy current $\langle J \rangle$ is zero.

Gordon equation. This agreement is not surprising at all since Eq. (14) is exact. The time-averaged energy current $\langle J \rangle$ is zero.

It is important to emphasize that the result in this comment is not applicable when the external field not only depends on t but also on x. In that case Eq. (10) cannot be obtained and, in principle, it is possible to observe a non-vanishing time-averaged energy current. A field of this kind has been considered in Ref. [1], where $f(x,t) = E(t) + \xi(x,t)$, with $E(t)$ being an ac field with zero mean and $\xi(x,t)$ a Gaussian white noise.

Acknowledgments

We acknowledge financial support from the Ministerio de Ciencia y Tecnología of Spain under the grants BFM2001-3878 (N.R.Q), BFM2003-03015 (B.S-R), and BFM2002-03822 (J.C-P), and from la Junta de Andalucía.

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